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Radiation from vortex filament motion near a half plane

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The two-dimensional low Mach number sound field from the passage of a line vortex around the edge of a rigid half plane is calculated by singular perturbation methods. Simple exact expressions are given for the space–time variation of the acoustic field, and for the total radiated energy and its frequency spectrum. These expressions show in particular that the effect of the half plane is to increase the energy radiated by a nearby vortical flow by M^{-3}, where M is a characteristic Mach number.

1. Introduction

Exact solutions for unsteady compressible rotational flow are very difficult to find. From the point of view of aerodynamic noise theory, only the solution given by Obermeier (1967) for the low Mach number sound field of a pair of spinning vortices appears to be of any significance as an example in which boundary effects are absent. Attempts to include the effects of solid surfaces in the flow increase the difficulty. Stüber (1970) and Rahman (1971) have solved one such problem including a solid boundary; that of the sound field generated when a pair of vortex filaments move under the influence of their images towards an infinite rigid plane. However, the surface effect there is trivial, involving merely a reflexion of sound by the plane. Another problem which is easily solved is that of the sound field from the rotation of a line vortex around a circular cylinder, provided the diameter of the circle described by the vortex is small compared with the appropriate acoustic wavelength. This problem can be simply and efficiently solved using matched asymptotic expansions. Again, however, the boundary effect is almost trivial. The cylinder scatters a dipole field and the intensity of the scattered sound varies as the fifth power of a typical flow velocity, which is just the analogue for two dimensions of Curle's (1955) prediction as to the effect of solid surfaces.

Now the general theories of aerodynamic noise scattering by flow–surface interaction (Ffowcs Williams & Hall 1970; Crighton & Leppington 1970, 1971) suggest that much more interesting and powerful effects occur if a flow is coupled to a large but inhomogeneous solid surface and it is important that these theories receive detailed confirmation from examination of whichever particular cases are susceptible to analysis. The author (Crighton 1972), following the work of Orszag & Crow (1970), has done this for the case of a vortex sheet leaving a semi-infinite plate and undergoing a spatial Kelvin–Helmholtz instability. This note aims to provide a further example of the sound field from a weakly compressible flow, with concentrated voticity, coupled to a large but inhomogeneous solid surface.

In the problem to be considered here, we again concern ourselves with a rigid half plane. A line vortex with its axis parallel to the edge of the plane is generated in the fluid surrounding the surface at a distance from the edge which is great compared with the shortest distance of the vortex from the plate. If the strength of the vortex is sufficiently small, and the circulation of the correct sense, the vortex moves with a speed which remains well subsonic and negotiates a path around the edge of the plate which is found in § 2. For most of the time, the vortex moves in a path which is essentially straight and thus radiates virtually no sound field. Energy radiation occurs only from the phase of the motion in which the vortex passes round the edge, and it turns out, rather obviously, that the acoustic wavelength set by the time scale for passage around the edge is large compared with the distance of the vortex from the edge during that time. Such a situation is clearly capable of singular perturbation treatment. The acoustic field can be found by matching a wave field, scaled on the wavelength, to an incompressible field scaled on a typical distance from vortex to edge.

Despite the lack of physical realizability of the flow, this example does enable us to derive simple expressions for all quantities of interest, at least to a first approximation in Mach number, and provides further confirmation of the important predictions of surface scattering theories.

2. Incompressible flow

Consider two-dimensional flow in the X, Y plane. A rigid semi-infinite plate lies in $X < 0$, $Y = 0$, and is surrounded by incompressible fluid. A line vortex with positive strength κ and axis parallel to the plate edge is generated above the plate at a great distance down from the edge. We find first the path of the vortex in the presence of the plate.

Define $X+iY = Z = R\exp i\theta$, $-\pi < \theta \leqslant +\pi$, then a simple mapping shows that at the instant in which the vortex lies at Z_0 the complex potential for the motion is

$$\Omega(Z) = -\frac{i\kappa}{2\pi}\ln\left(Z^{\frac{1}{2}} - Z_0^{\frac{1}{2}}\right) + \frac{i\kappa}{2\pi}\ln\left(Z^{\frac{1}{2}} + \bar{Z}_0^{\frac{1}{2}}\right), \qquad (2.1)$$

the overbar denoting the complex conjugate. The only singularity of $\Omega(Z)$ is at Z_0, for $Z^{\frac{1}{2}} + \bar{Z}_0^{\frac{1}{2}} \neq 0$ when $\arg Z$ and $\arg Z_0$ are defined to lie in $(-\pi, +\pi)$. The fluid velocity components at a general point are given by

$$u_x - iu_y = \Omega'(Z),$$

and in particular,

$$\Omega'(Z_0+\epsilon) = -\frac{i\kappa}{2\pi\epsilon} + \frac{i\kappa}{8\pi Z_0} + \frac{i\kappa}{4\pi}\frac{1}{Z_0 + |Z_0|} + O(\epsilon)$$

as $\epsilon \to 0$. The first term represents the motion induced directly by the vortex, which must move with the velocity field induced by all causes other than the vortex itself. Hence the equations for the path of the vortex are

$$\frac{dX_0}{dt} - i\frac{dY_0}{dt} = \frac{i\kappa}{8\pi Z_0} + \frac{i\kappa}{4\pi}\frac{1}{Z_0 + |Z_0|},$$

or equivalently,

$$dR_0/dt = -(\kappa/8\pi R_0)\tan\tfrac{1}{2}\theta_0, \quad d\theta_0/dt = -\kappa/4\pi R_0^2. \tag{2.2}$$

These equations have the solutions

$$\left.\begin{aligned} R_0 &= a\sec\tfrac{1}{2}\theta_0, \\ \theta_0 &= 2\tan^{-1}(-\kappa t/8\pi a^2), \\ R_0 &= \{a^2 + (\kappa t/8\pi a)^2\}^{\frac{1}{2}} \end{aligned}\right\} \tag{2.3}$$

and show that the vortex starts at $(-\infty, 2a)$ with velocity $(\kappa/8\pi a, 0)$ when $t = -\infty$, passing through the point $(a, 0)$ with velocity $(0, -\kappa/4\pi a)$ at $t = 0$ and proceeding back to $(-\infty, -2a)$ in a symmetrical fashion.

It is convenient to make the space co-ordinates dimensionless using the minimum distance a and to scale the time variable on a/U, where $U = \kappa/4\pi a$. Further, we normalize the complex potential on $2aU$ and then, using the same symbols as before, but referring now to the dimensionless variables, the incompressible flow problem is defined by the equations

$$\left.\begin{aligned} R_0 = \sec\tfrac{1}{2}\theta_0, \tan\tfrac{1}{2}\theta_0 &= -\tfrac{1}{2}t, \\ \Omega(Z) = -i\ln(Z^{\frac{1}{2}} - Z_0^{\frac{1}{2}}) &+ i\ln(Z^{\frac{1}{2}} + \bar{Z}_0^{\frac{1}{2}}). \end{aligned}\right\} \tag{2.4}$$

3. The sound field

Suppose now that the fluid is slightly compressible, in the sense that

$$M = U/a_0 \ll 1,$$

a_0 being the uniform sound speed. As remarked in §1, acoustic radiation takes place only over the time a/U during which the vortex undergoes appreciable acceleration. In this time, acoustic influence extends over a roughly circular region of radius aM^{-1} centred on the plate edge. This suggests that we rescale *both* space co-ordinates on aM^{-1}, defining $(x, y) = (MX, MY)$ but leaving the time scaled on a/U as before. The implications of the two sets of dimensionless co-ordinates for the equations of motion in the limit $M \to 0$ have been examined in detail by Obermeier (1967) and Rahman (1971). Here we merely state the obvious consequence, that for fixed X, Y and $M \to 0$ we obtain Laplace's equation

$$\nabla_X^2 \phi = 0$$

for the velocity potential, while for fixed x, y and $M \to 0$ we have

$$\left(\frac{\partial^2}{\partial t^2} - \nabla_x^2\right)\phi = 0.$$

For both cases we have to impose the condition that ϕ has zero normal derivative on the half plane, while for the outer wave-field solution a radiation condition must apply at infinity.

To find the appropriate form of outer potential we express (2.4) in outer coordinates x, y, expand for $M \to 0$, and take the real part to obtain

$$\phi(x/M, y/M, t) = (2R_0^{\frac{1}{2}} \cos \tfrac{1}{2}\theta_0) \frac{M^{\frac{1}{2}} \sin \frac{1}{2}\theta}{r^{\frac{1}{2}}} + O(M)$$

$$= \frac{4^{\frac{3}{4}}}{(t^2+4)^{\frac{1}{4}}} \frac{M^{\frac{1}{2}} \sin \frac{1}{2}\theta}{r^{\frac{1}{2}}} + O(M)$$

in view of the equation (2.4) for the vortex path. This suggests that the potential in the wave field has the form

$$\phi(x, y, t) = M^{\frac{1}{2}} f(x, y, t) + o(M^{\frac{1}{2}}),$$

where f satisfies the wave equation, the boundary condition on the plate, a radiation condition at infinity and, in order to match the inner incompressible solution, is such that

$$\lim_{M \to 0} \{M^{\frac{1}{2}} f(MX, MY, t)\} = \frac{4^{\frac{3}{4}}}{(t^2+4)^{\frac{1}{4}}} \frac{\sin \frac{1}{2}\theta}{R^{\frac{1}{2}}}. \tag{3.1}$$

The simplest way of finding f is to take Fourier transforms in time, defining

$$f(x, y, \omega) = \int_{-\infty}^{+\infty} f(x, y, t) \exp(i\omega t)\, dt.$$

Then
$$(\nabla_{\mathbf{x}}^2 + \omega^2) f(x, y, \omega) = 0 \tag{3.2}$$

and the matching condition (3.1) is

$$\lim_{M \to 0} \{M^{\frac{1}{2}} f(MX, MY, \omega)\} = \int_{-\infty}^{+\infty} \frac{4^{\frac{3}{4}}}{(t^2+4)^{\frac{1}{4}}} \frac{\sin \frac{1}{2}\theta}{R^{\frac{1}{2}}} \exp(i\omega t)\, dt$$

$$= \frac{8}{(2\pi)^{\frac{1}{2}}} \Gamma(\tfrac{3}{4})\, |\omega|^{-\frac{1}{4}} K_{-\frac{1}{4}}(2|\omega|) \frac{\sin \frac{1}{2}\theta}{R^{\frac{1}{2}}}, \tag{3.3}$$

where $K_{-\frac{1}{4}}$ denotes the modified Bessel function of the second kind and of order $-\frac{1}{4}$ and the value of the integral is taken from Gradshteyn & Ryzhik (1965, p. 426). Evidently it is sufficient to consider only positive frequencies ω. Then it is easy to prove that the general solution for $f(x, y, \omega)$, subject to the requirements that $\partial f/\partial y = 0$ on $\theta = \pm\pi$ and that $f(x, y, \omega) \exp(-i\omega t)$ shall represent an outgoing wave field at infinity, is

$$f(x, y, \omega) = \sum_{n=0}^{\infty} A_n(\omega) H_{n+\frac{1}{2}}^{(1)}(\omega r) \sin(n+\tfrac{1}{2})\,\theta + \sum_{n=0}^{\infty} B_n(\omega) H_n^{(1)}(\omega r) \cos n\theta.$$

Only one term of this series is capable of satisfying the matching requirement (3.3) and we find immediately that

$$f(x, y, \omega) = A(\omega) H_{\frac{1}{2}}^{(1)}(\omega r) \sin \tfrac{1}{2}\theta, \tag{3.4}$$

where
$$A(\omega) = 4i\Gamma(\tfrac{3}{4})\, \omega^{\frac{1}{4}} K_{-\frac{1}{4}}(2\omega). \tag{3.5}$$

For negative frequencies we use the fact that $f(x, y, -\omega) = \bar{f}(x, y, \omega)$.

The expressions (3.4) and (3.5) are useful for constructing the frequency spectrum of the total radiated energy. It is possible also, however, to invert the

Fourier transform and to find the wave-field potential as a function of $\mathbf{x}$ and t. We have

$$\phi(x,y,t) = \frac{M^{\frac{1}{2}}\sin\frac{1}{2}\theta}{\pi}\,\mathrm{Re}\int_0^{+\infty} A(\omega)\,H^{(1)}_{\frac{1}{2}}(\omega r)\exp(-i\omega t)\,d\omega$$

$$= \frac{2M^{\frac{1}{2}}\sin\frac{1}{2}\theta}{\pi}\,\Gamma(\tfrac{3}{4})\left(\frac{2}{\pi r}\right)^{\frac{1}{2}}\int_0^{\infty}\omega^{-\frac{1}{4}}K_{-\frac{1}{4}}(2\omega)\cos\omega(r-t)\,d\omega.$$

The integral is tabulated (Gradshteyn & Ryzhik 1965, p. 749) and with use of the formula $\Gamma(x)\,\Gamma(1-x) = \pi\operatorname{cosec}\pi x$ gives

$$\phi(x,y,t) = \frac{4^{\frac{3}{4}}}{\{(r-t)^2+4\}^{\frac{1}{4}}}\,\frac{M^{\frac{1}{2}}\sin\frac{1}{2}\theta}{r^{\frac{1}{2}}}. \tag{3.6}$$

This expression obviously matches the incompressible flow solution in the manner demanded by (3.1). We see that, near the wave front $r = t$, the potential has the form $r^{-\frac{1}{2}}\sin\frac{1}{2}\theta$ typical of rigid half-plane problems. This behaviour is also found in each individual frequency component, as shown by (3.4). The maximum potential at any point (r,θ) is reached at a time r after the vortex passes through the point $(a, 0)$ on the extension of the half plane. For fixed t, $\phi \sim r^{-1}$ as $r \to \infty$ while for fixed r, $\phi \sim |t|^{-\frac{1}{2}}$ as $|t| \to \infty$, so that the field varies remarkably slowly.

The total energy radiated by the vortex throughout its history is most easily found from (3.4) and (3.5). After a little algebra we find

$$E = \frac{4^{\frac{3}{2}}\Gamma^2(\frac{3}{4})}{\pi}\rho_0\,U^2Ma^2\int_0^{\infty}\mathscr{E}(\omega)\,d\omega, \tag{3.7}$$

where the spectral distribution of energy over the (dimensionless) frequency ω is given by

$$\mathscr{E}(\omega) = \omega^{\frac{3}{2}}K^2_{-\frac{1}{4}}(2\omega). \tag{3.8}$$

The (constant) value of the integral in (3.7) can be found from Gradshteyn & Ryzhik (1965, p. 693) but is of no significance here.

The interesting point of (3.7) is the variation of E with the third power of velocity. This is to be contrasted with the U^6 variation found by Obermeier (1967), Stüber (1970) and Rahman (1971) for the radiated energy of vortex filament systems in free space or in the presence of an *infinite* rigid wall. This represents a confirmation of general theories (Ffowcs Williams & Hall 1970; Crighton & Leppington 1970, 1971) of the scattering from unsteady flow interaction with a half plane on two counts. First, we have the directivity function $\sin\frac{1}{2}\theta$, and second, the prediction that the effect of flow interaction with a rigid half plane is to increase the power output of the flow by M^{-3}. The spectral distribution of energy has just the features which might be expected. $\mathscr{E}(\omega)$ behaves like ω as $\omega \to 0$ and like $\omega^{\frac{1}{2}}\exp(-4\omega)$ as $\omega \to \infty$, with a single maximum at a dimensionless frequency around unity.

4. Conclusions

The method of matched asymptotic expansions provides an efficient means of calculating the radiation from low Mach number flows generated by systems of vortex filaments. In the past, it has been applied by workers at Göttingen

(Obermeier and Rahman) to flows in free space or in the presence of an infinite plane boundary It is possible to treat those problems using Lighthill's (1952) general theory together with Powell's (1964) expression for Lighthill's quadrupole inhomogeneity, and Stüber (1970) has confirmed the results of Obermeier and Rahman in this way.

In the present note we have applied the singular perturbation method to a vortex filament coupled to a rigid half plane. Simple expressions, (3.6), (3.7) and (3.8), are readily derived for all quantities of interest and the results confirm the predictions of general theories as to the great scattering efficiency of the half plane. Lighthill's theory might also be used to obtain these results provided one were to evaluate Lighthill's integral using the *exact* Green's function for a rigid half plane rather than the free-space Green's function. Of course, it would also be advisable to use Powell's expression div $(\mathbf{u} \times \boldsymbol{\omega})$, where $\boldsymbol{\omega}$ now denotes the vorticity in place of Lighthill's $\partial^2 T_{ij}/\partial x_i \partial x_j$, in order to take advantage of the delta function singularity of the vorticity in the present problem. There are considerable difficulties in such an approach, however, and the additional disadvantage of this method in requiring at least a suitable approximation to the exact Green's function turns the scales heavily in favour of the matching approach.

The more general problem of the passage of a line vortex around the edge of any wedge with included angle a rational fraction of π can also be solved by the perturbation method. The general features of the solution – i.e. the directivity and the velocity exponent for the radiated energy – follow at once from simple consideration of the mapping used to solve the incompressible problem and emerge so obviously as to need no further comment.

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